Electro-hydrodynamics
(Theory of electrostatic spinning)

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Table of contents:

Chapter 1:

Ideal fluids

1.1 The equation of continuity

When we start to thing about fluids, after while we'll reach the problem of its description by standard way. We cannot describe them as some piece of matter, but we have to use the term of continuum instead. Continuum is the term which can be translated as uninterrupted connection. This means, that when we look deeply into the fluids, we'll see many small particles moving in all possible directions. So when we try to describe them by movements of all this small particles, very quickly we'll reach the problem with definition of velocity field and the system will be very difficult to solve without powerful computers. Due to the effort of finding their fundamental description, the theory of continuity was implemented. This means that we have to look at these fluids as a system consist of many small elementary volumes. The speed of fluid does not belong to these elementary volumes but their particles. So continuum is a mixture of elementary volumes containing constantly moving particles and we have to focus on these elementary volumes and observe what is going on with particles inside. Let have one of this volume and look at its flow (Fig.1.1).

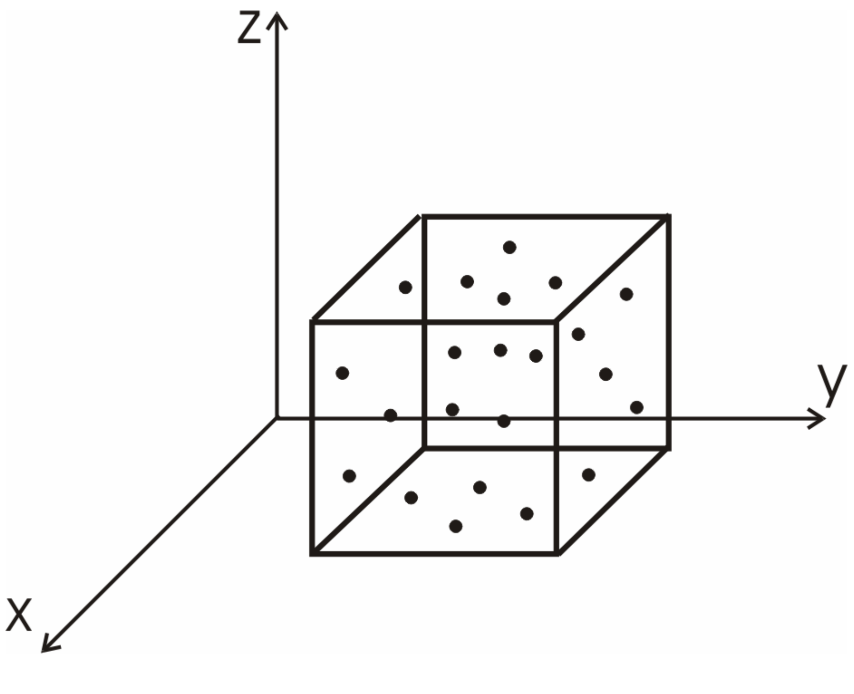


Fig. 1.1: Elementary volume containing constantly moving particles. This volume has to be infinitesimally small compare to the size of the particles.

The total mass of fluid flowing out of the volume V_0 in unit time, where the integration is taken over the whole of the closed surface surrounding the volume, versus decrease per unit time t in the mass of fluid in the volume V_0 can be written:

$$\frac{\partial}{\partial t}\int \rho\vec{v}d\vec{f} = -\oint div(\rho\vec{v})dV \tag{1.1}$$

Where ρ is mass density, $\vec{v}$ is a velocity of the fluid and $d\vec{f}$ is vector of the area of the surface element. If we apply Green's / Gauss formula, one can get:

$$\int[\frac{\partial\rho}{\partial t} + div(\rho\vec{v})]dV = 0 \tag{1.2}$$

Where

$$\frac{\partial\rho}{\partial t} + div(\rho\vec{v}) = 0 \tag{1.3}$$

is the equation of continuity and if we consider incompressible fluids $\rho = const$, one can get following formula:

$$div(\vec{v}) = 0 \tag{1.4}$$

1.2. <u>Euler's equation</u>

We start to thing about total force acting on volume:

$$-\oint p d\vec{f} = -\int \vec{\nabla} p dV \tag{1.5}$$

where p is Pascal's omni directional pressure. Comparing with Newton's formula

$$\frac{d\vec{v}}{dt} = -\frac{1}{\rho}\vec{\nabla}p \tag{1.6}$$

But it is almost impossible to find global derivation of velocity, because in continuous media everything has been changing per unit of time $\vec{v}[x(t), y(t), z(t), t]$. So therefore:

$$\frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v} + \frac{1}{\rho}\vec{\nabla}p = 0 \tag{1.7}$$

This is the Euler's equation. One can see that $\vec{v} = A/T$ where T is a period of the wave, A is its amplitude, λ is its wavelength. We get: $\vec{v}\cdot\vec{\nabla} \cong A/(T\lambda)$ and $(\vec{v}\cdot\vec{\nabla})\vec{v} \cong A^2/(T^2\lambda)$. Because for our needs the amplitude of the wave is much lower than its wavelength $A >> \lambda$ the expression $(\vec{v}\cdot\vec{\nabla})\vec{v}$ can be neglected. One can reach following formula:

$$\frac{\partial\vec{v}}{\partial t} + \frac{1}{\rho}\vec{\nabla}p = 0 \tag{1.8}$$

By applying of curl on the both sides:

$$\vec{\nabla} \times \frac{\partial \vec{v}}{\partial t} + \frac{1}{\rho} \vec{\nabla} \times (\vec{\nabla} p) = 0 \qquad (1.9)$$

By the definition: $\vec{\nabla} \times (\vec{\nabla} p) \equiv 0$ therefore: $\vec{\nabla} \times \frac{\partial \vec{v}}{\partial t} = 0$ and $\vec{\nabla} \times \vec{v} = const$

One can now implement velocity potential by: $\vec{v} = \vec{\nabla}\Phi$ and after comparison with formula (1.4) we get: $$\Delta\Phi = 0 \qquad (1.10)$$

Formula (1.8) can now be written as :

$$\vec{\nabla}\left(\frac{\partial}{\partial t}\Phi + \frac{1}{\rho} p\right) = 0 \qquad (1.11)$$

1.3 Gravity waves with infinity depth

Equation (1.10) has many solutions and to solve it, one has to propose some solution for Φ. We have to find some limitations if we want to solve it. First of all, let's consider only uncompressible fluids: $\rho = const$, let's use Laplace's formula (1.10) and let's use boundary condition for infinity depth: $\lim_{z \to -\infty} \phi = 0$. Because surface of the liquid has been made of many small waves with small amplitudes compare to its wavelengths, the velocity potential will be wave function. Let consider two dimensional wave function and let dimension y be constant. The solution will be

$$\Phi = A \cdot \exp(kz) \cdot \cos(k\vec{r} - \omega t) \qquad (1.12)$$

Using another boundary condition and consider only hydrostatic pressure, the formula (1.11) will for our purpose be:

$$\frac{\partial^2\Phi}{\partial t^2} + g\frac{\partial\zeta}{\partial t} = 0 \qquad (1.13)$$

Where ζ is the vertical displacement of the surface in its oscillations and $\partial\zeta/\partial t = v_z$ is the vertical component of the velocity and is equal to:

$$\frac{\partial\zeta}{\partial t} = \frac{\partial\Phi}{\partial z} \qquad (1.14)$$

By assignment of velocity potential (1.11) to (1.13) with respect of (1.14) we get dispersion law for gravity waves: $$\omega^2 = kg \qquad (1.15)$$

Knowledge of dispersion law plays crucial role in description of liquid surface, which is important for electrospinning.

1.4 Gravity waves with depth h

Now let's thing about gravity waves with length large compared with depth of the liquid. We consider situations with very tiny liquid layers as for example is case of nanospider. Boundary condition of such system is, that the normal velocity component at the bottom of the liquid, must be zero, i.e.

$$v_z = \left.\frac{\partial \Phi}{\partial z}\right|_{z=-h} = 0 \qquad (1.16)$$

General solution of this system is:

$$\Phi = [A \cdot \exp(kz) + B \cdot \exp(-kz)] \cdot \exp[i(kx - \omega t)] \qquad (1.17)$$

With boundary condition mentioned above one can find ratio between constants A and B:

$$\Phi = A \cdot \cosh[k(z + h)] \cdot \exp[i(kx - \omega t)] \qquad (1.18)$$

Now let's find solution for deflection of the wave:

$$\zeta = C \cdot \exp[i(kx - \omega t)] \qquad (1.19)$$

Another condition is that velocity in z direction must be equal to deflection change $\partial\zeta$ in time.

$$v_z = \left.\frac{\partial \Phi}{\partial z}\right|_{z=\zeta\cong 0} = \frac{\partial \zeta}{\partial t} \qquad (1.20)$$

After constitution of (1.18) and (1.19) to (1.20) one can find ratio between constants:

$$A = \frac{-iC\omega}{k \cdot \sinh(kh)} \qquad (1.21)$$

Now we can use Euler's formula in following form:

$$\frac{\partial}{\partial t}\Phi + \frac{1}{\rho} p = 0 \qquad (1.22)$$

With only hydrostatic pressure we can find relation between ω and k :

$$\omega^2 = kg \cdot \tanh(kh) \qquad (1.23)$$

It can be clearly seen, that with increasing of depth h, the $\tanh(kh)$ will get closer to 1 and both results (1.15) and (1.23) will merge together.

1.5 Capillary pressure

Now let's think about curved surface of the liquid in two dimensions x, z and let y be unchangeable. There is another form of pressure in real liquid called capillary pressure, but only its vertical direction is important for electrospinning. $p_c = F_c/S = \gamma_0 L/S$, where γ_0 is a vertical surface tension and L is length of the wave in y direction (see Fig. 1.2 A).

One can see in Fig.1.2 B that $\gamma_0 = \gamma \cdot \sin(d\varphi)$ and $S = L \cdot 2d\varphi \cdot R$. For very small angles can be $\sin(d\varphi)$ approximated by $d\varphi$. So now we find well known formula for capillary pressure of tube with one radius of curvature:

$$p_c = \frac{\gamma}{R} \tag{1.24}$$

Formula (1.24) can be rewritten for ball: $p_c = 2\gamma/R$ and for sphere: $p_c = \gamma(1/R_1 + 1/R_2)$. But we are not interested in capillary pressure as a function of radius of curvature of the wave, but as a function of its depth $\zeta(x)$. There is axiom, that arbitrary curve can be approximated by series of circles (see in Fig. 1.2 C) The radius of curvature can be rewritten as:

$$\frac{1}{R} = \lim_{ds \to 0} \frac{d\alpha}{R \cdot d\alpha} \tag{1.25}$$

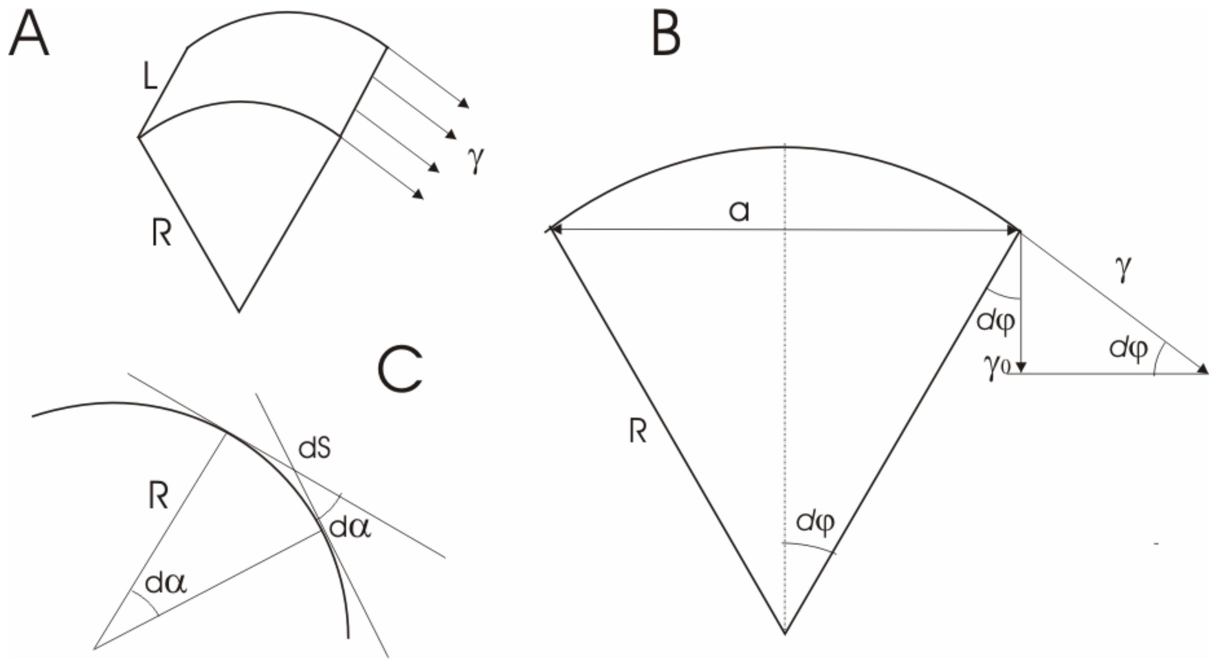
Now we can write: $d\zeta / dx = tg(\alpha)$ and for small angles: $tg(\alpha) \cong \alpha$. But we are interested in change of this angle in x direction. So:

$$\lim_{ds \to 0} \frac{\alpha(x+dx) - \alpha(x)}{(x+dx) - x} = \frac{d\alpha}{dx} \tag{1.26}$$

From (1.25) and (1.26) we can write: $\frac{1}{R} \cong \frac{\partial^2 \zeta}{\partial x^2}$. So finally for capillary pressure we have important formula:

$$p_c = -\gamma \frac{\partial^2 \zeta}{\partial x^2} \tag{1.27}$$

Capillary pressure is always negative due its actuation on the surface, which is always against deflection. So when the deflection is positive, capillary pressure has to be negative and vice versa. This is important to keep surface of the liquid stabilized.



1.6. Capillary waves

In previous item we found very important formula for capillary pressure. With this knowledge, one can search for dispersion law included this pressure. We'll do it for both examples of infinity and tiny depths. First of all we have to include capillary pressure (1.27) to formula (1.12). Euler's formula will be:

$$\frac{\partial \Phi}{\partial t} + g\zeta(x) - \gamma \frac{\partial^2 \zeta}{\partial x^2} = 0 \qquad (1.28)$$

With derivation by the time, using (1.14) and using of proposed solution of Laplace's formula (1.11) we get dispersion law:

$$\omega^2 = kg \cdot + \gamma k^3 / \rho \qquad (1.29)$$

Do the same for thin film and following chapter 1.4 with capillary pressure, using boundary condition (1.16) and general solution of Laplace's formula (1.17),(1,18), one can get dispersion law in following form:

$$\omega^2 = (g\rho \cdot + \gamma k^2) \frac{k}{\rho} \tanh(kh) \qquad (1.30)$$